HADAMARD TYPE INEQUALITIES FOR S-GEO-CONVEX FUNCTIONS

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Abstract. In this paper, the authors establish some new Hadamard type inequalities using elementary well-known inequalities for functions whose first derivatives absolute values are geometrically and s-geometrically convex, which are given below respectively as

$$f\left(x^t y^{1-t}\right) \le \left[f(x)\right]^t \left[f(y)\right]^{1-t}$$

and

$$f\left(x^t y^{1-t}\right) \le \left[f(x)\right]^{t^s} \left[f(y)\right]^{(1-t)^s}$$

where $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$, for some fixed $s \in (0,1]$, $x, y \in I \subset \mathbb{R}_+$ and $t \in [0,1]$. And some applications to special means for positive numbers are given.

Keywords: Geometrically convex, Hadamard's inequality, $s-$geometrically convex, Special means.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I f real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

hold. This double inequalities are known in the literature as the Hermite-Hadamard inequality for convex functions.

In recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see [3-5,7,8,10].

In this section we will present definitions and some results used in this paper.

Definition 1. Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}, \varnothing \ne I \subseteq \mathbb{R}$ is said to be convex if

$$f\left(tx + (1-t)y\right) \le tf(x) + (1-t)f(y). \qquad (1)$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 2. [6] Let $s \in (0,1]$. A function $f : I \subset \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be $s-$ convex in the second sense if

$$f\left(tx+(1-t)y\right) \le t^s f(x) + (1-t)^s f(y) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily checked for $s=1$, $s-$convexity reduces to the ordinary convexity of functions defined on [0,∞).

Recently, [9], the concept of geometrically and $s-$geometrically convex functions were introduced as follows.

Definition 3. [9], A function $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^t y^{1-t}\right) \le \left[f(x)\right]^t \left[f(y)\right]^{1-t} \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 4. [9], A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $s-$geometrically convex function if

$$f\left(x^t y^{1-t}\right) \le \left[f(x)\right]^{t^s} \left[f(y)\right]^{(1-t)^s} \tag{4}$$

for some $s \in (0,1]$, where $x, y \in I$ and $t \in [0,1]$.

If $s=1$, the $s-$geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

Example 1. [9],Let $f(x) = x^s / s,\ x \in (0,1],\ 0 < s < 1,\ q \ge 1,$ and then the function

$$\left|f'(x)\right|^q = x^{(s-1)q} \tag{5}$$

is monotonically decreasing on $(0,1]$. For $t \in [0,1]$, we have

$$(s-1)q\left(t^s - t\right) \le 0,\ (s-1)q\left((1-t)^s - (1-t)\right) \le 0. \tag{6}$$

Hence, $\left|f'(x)\right|^q$ is $s-$geometrically convex on $(0,1]$ for $0 < s < 1$.

2. Some Hadamard Type Inequalities

In order to prove our main theorems, we need the following lemma 1.

Lemma 1. [1], Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{b-a}{4}\left[\int_0^1(-t)\,f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right.\\&\qquad\left.+\int_0^1(t)\,f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right].\end{aligned}\tag{7}$$

A simple proof of this equality can be also done integrating by parts in the right hand side. The details are left to the interested reader.

Lemma 2. [2], *If* $0<\phi\leq 1\leq\mu,\ 0<\alpha,\beta\leq 1,$ then

$$\phi^{\alpha^{\beta}}\leq\phi^{\alpha\beta}\ \text{ and }\ \mu^{\alpha^{\beta}}\leq\mu^{\beta\alpha+1-\beta}.\tag{8}$$

The next theorems give a new result of the upper Hermite-Hadamard inequality for $s-$geometrically convex functions.

In the following part of the paper;

$$\alpha(u,v)=\left|f'(a)\right|^{u}\left|f'(b)\right|^{-v},u,v\geq 0,\tag{9}$$

$$g_1(\alpha)=\begin{cases}\dfrac{1}{2} & \alpha=1\\ \dfrac{\alpha\ln\alpha-\alpha+1}{(\ln\alpha)^2} & \alpha\neq 1,\end{cases}\tag{10}$$

and

$$g_2(\alpha)=\begin{cases}1 & \alpha=1\\ \dfrac{\alpha-1}{\ln\alpha} & \alpha\neq 1.\end{cases}\tag{11}$$

Theorem 1. Let $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ be differentiable mapping on $I^{\circ},a,b\in I$ with $a<b$ and f' is integrable on $[a,b]$. If $|f'|$ is $s-$geometrically convex and monotonically decreasing on $[a,b]$ and $s\in(0,1]$, then the following inequality holds

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\begin{cases}\dfrac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left(g_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+g_1\left(\alpha\left(\frac{-s}{2},\frac{-s}{2}\right)\right)\right)\\ \qquad\qquad if\ 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1,\\ \dfrac{b-a}{4}\left|f'(a)\right|^{\frac{2-s}{2}}\left|f'(b)\right|^{\frac{s}{2}}\left(g_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+g_1\left(\alpha\left(\frac{-s}{2},\frac{-s}{2}\right)\right)\right)\\ \qquad\qquad if\ \left|f'(b)\right|\leq 1\leq\left|f'(a)\right|,\\ \dfrac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{2-s}{2}}\left(g_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+g_1\left(\alpha\left(\frac{-s}{2},\frac{-s}{2}\right)\right)\right)\\ \qquad\qquad if\ 1\leq\left|f'(a)\right|,\left|f'(b)\right|.\end{cases}\end{aligned}\tag{12}$$

Proof. Since $|f'|$ is an $s-$geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1 we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$
$$\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|dt+\int_0^1|t|\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|dt\right]$$
$$\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'(a)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(b)\right|^{\left(\frac{1-t}{2}\right)^s}dt+\int_0^1|t|\left|f'(b)\right|^{\left(\frac{1+t}{2}\right)^s}\left|f'(a)\right|^{\left(\frac{1-t}{2}\right)^s}dt\right].$$

By (8), when $0<|f'(a)|,|f'(b)|\le 1$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'(a)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)}dt+\int_0^1|t|\left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)}dt\right]$$
$$= \frac{b-a}{4}\left|f'(a)\,f'(b)\right|^{\frac{s}{2}}\left[\int_0^1|-t|\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2}}dt+\int_0^1|t|\left|\frac{f'(b)}{f'(a)}\right|^{\frac{st}{2}}dt\right]$$
$$= \frac{b-a}{4}\left|f'(a)\,f'(b)\right|^{\frac{s}{2}}\left(g_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+g_1\left(\alpha\left(\frac{-s}{2},\frac{-s}{2}\right)\right)\right),$$

when $|f'(b)|\le 1\le|f'(a)|$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'(a)\right|^{s\left(\frac{1+t}{2}\right)+1-s}\left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)}dt+\int_0^1|t|\left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)}\left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)+1-s}dt\right]$$
$$= \frac{b-a}{4}\left|f'(a)\right|^{\frac{2-s}{2}}\left|f'(b)\right|^{\frac{s}{2}}\left[\int_0^1|-t|\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2}}dt+\int_0^1|t|\left|\frac{f'(b)}{f'(a)}\right|^{\frac{st}{2}}dt\right]$$
$$= \frac{b-a}{4}\left|f'(a)\right|^{\frac{2-s}{2}}\left|f'(b)\right|^{\frac{s}{2}}\left(g_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+g_1\left(\alpha\left(\frac{-s}{2},\frac{-s}{2}\right)\right)\right),$$

when $1\le|f'(a)|,|f'(b)|$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'(a)\right|^{s\left(\frac{1+t}{2}\right)+1-s}\left|f'(b)\right|^{s\left(\frac{1-t}{2}\right)+1-s}dt+\int_0^1 |t|\left|f'(b)\right|^{s\left(\frac{1+t}{2}\right)+1-s}\left|f'(a)\right|^{s\left(\frac{1-t}{2}\right)+1-s}dt\right]$$
$$=\frac{b-a}{4}\left|f'(a)\,f'(b)\right|^{\frac{2-s}{2}}\left[\int_0^1 |-t|\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2}}dt+\int_0^1 |t|\left|\frac{f'(b)}{f'(a)}\right|^{\frac{st}{2}}dt\right]$$
$$=\frac{b-a}{4}\left|f'(a)\,f'(b)\right|^{\frac{2-s}{2}}\left(g_1\left(\alpha\left(\frac{s}{2},\frac{s}{2}\right)\right)+g_1\left(\alpha\left(\frac{-s}{2},\frac{-s}{2}\right)\right)\right)$$

which completes the proof.

Theorem 2. Let $f:I\subset\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, $a,b\in I$, with $a<b$ and $f'\in L([a,b])$. If $|f'|^q$ is $s-$geometrically convex and monotonically decreasing on $[a,b]$ for $p,q>1$ and $s\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\times$$
$$\begin{cases}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left(\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right) & \\ \qquad\qquad if\ \ 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1, & \\ \left|f'(a)\right|^{\frac{2-s}{2}}\left|f'(b)\right|^{\frac{s}{2}}\left(\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right) & \\ \qquad\qquad if\ \ \left|f'(b)\right|\leq 1\leq\left|f'(a)\right|, & \\ \left|f'(a)f'(b)\right|^{\frac{2-s}{2}}\left(\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right) & \\ \qquad\qquad if\ 1\leq\left|f'(a)\right|,\left|f'(b)\right|. & \end{cases} \tag{13}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f'|^q$ is an $s-$geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1 and well-known Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$
$$\leq \frac{b-a}{4}\left[\left[\int_0^1 t^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}+\left[\int_0^1 t^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left[\int_0^1\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}+\left[\int_0^1\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left[\int_0^1\left(|f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}\right)^q dt\right]^{\frac{1}{q}}+\left[\int_0^1\left(|f'(b)|^{\left(\frac{1+t}{2}\right)^s}|f'(a)|^{\left(\frac{1-t}{2}\right)^s}\right)^q dt\right]^{\frac{1}{q}}\right].$$

By (8), when $0<|f'(a)|,|f'(b)|\leq 1$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left[\int_0^1\left(|f'(a)|^{sq\left(\frac{1+t}{2}\right)}|f'(b)|^{sq\left(\frac{1-t}{2}\right)}\right)dt\right]^{\frac{1}{q}}+\left[\int_0^1\left(|f'(b)|^{sq\left(\frac{1+t}{2}\right)}|f'(a)|^{sq\left(\frac{1-t}{2}\right)}\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{\frac{s}{2}}\left[\left(\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{\frac{s}{2}}\left(\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

when $|f'(b)|\leq 1\leq |f'(a)|$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left[\int_0^1\left(|f'(a)|^{sq\left(\frac{1+t}{2}\right)+q(1-s)}|f'(b)|^{sq\left(\frac{1-t}{2}\right)}\right)dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1\left(|f'(b)|^{sq\left(\frac{1+t}{2}\right)}|f'(a)|^{sq\left(\frac{1-t}{2}\right)+q(1-s)}\right)dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}|f'(a)|^{1-\frac{s}{2}}|f'(b)|^{\frac{s}{2}}\left[\left(\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}|f'(a)|^{\frac{2-s}{2}}|f'(b)|^{\frac{s}{2}}\left(\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

when $1\leq|f'(a)|,|f'(b)|,$ we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left[\int_0^1\left(|f'(a)|^{sq\left(\frac{1+t}{2}\right)+q(1-s)}|f'(b)|^{sq\left(\frac{1-t}{2}\right)+q(1-s)}\right)dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1\left(|f'(b)|^{sq\left(\frac{1+t}{2}\right)+q(1-s)}|f'(a)|^{sq\left(\frac{1-t}{2}\right)+q(1-s)}\right)dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{1-\frac{s}{2}}\left[\left(\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|\frac{f'(b)}{f'(a)}\right|^{\frac{sq}{2}t}dt\right)^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4(p+1)^{\frac{1}{p}}}|f'(a)f'(b)|^{1-\frac{s}{2}}\left(\left[g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

which completes the proof.

Corollary 1. Let $f: I\subseteq(0,\infty)\to(0,\infty)$ be differentiable on I°, $a,b\in I$ with $a<b$ and $f'\in L([a,b])$. If $|f'|^{q}$ is $s-$geometrically convex and monotonically decreasing on $[a,b]$ for $p,q>1$ and $s\in(0,1]$, then

$i)$ When $p=q=2$, then one has

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| = \frac{b-a}{4\sqrt{3}}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left(\sqrt{g_2(\alpha(s,s))}+\sqrt{g_2(\alpha(-s,-s))}\right),$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\le 1$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$=\frac{b-a}{4\sqrt{3}}\left|f'(a)\right|^{1-\frac{s}{2}}\left|f'(b)\right|^{\frac{s}{2}}\times\left(\sqrt{g_2(\alpha(s,s))}+\sqrt{g_2(\alpha(-s,-s))}\right),$$

where $\left|f'(b)\right|\le 1\le\left|f'(a)\right|$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| = \frac{b-a}{4\sqrt{3}}\left|f'(a)f'(b)\right|^{1-\frac{s}{2}}\left(\sqrt{g_2(\alpha(s,s))}+\sqrt{g_2(\alpha(-s,-s))}\right),$$

where $1\le\left|f'(a)\right|,\left|f'(b)\right|$.

ii) If we take $s=1$ in (13), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left|f'(a)f'(b)\right|^{\frac{1}{2}}\left(\left[g_2\left(\alpha\left(\frac{q}{2},\frac{q}{2}\right)\right)\right]^{\frac{1}{q}}+\left[g_2\left(\alpha\left(\frac{-q}{2},\frac{-q}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\le 1$.

Theorem 3. Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, $a,b\in I$, with $a<b$ and $f'\in L([a,b])$. If $\left|f'\right|^q$ is $s-$geometrically convex and monotonically decreasing on $[a,b]$ for $q\ge 1$ and $s\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\times$$

$$\begin{cases}\left(\mu^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right) & \\ \qquad if\ 0<\left|f'(a)\right|,\left|f'(b)\right|\le 1, \\ \left|f'(a)\right|^{1-s}\left(\mu^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right) & \\ \qquad if\ \left|f'(b)\right|\le 1\le\left|f'(a)\right|, \\ \left|f'(a)f'(b)\right|^{1-s}\left(\mu^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right) & \\ \qquad if\ 1\le\left|f'(a)\right|,\left|f'(b)\right|. \end{cases} \tag{14}$$

where $\mu=\left|\frac{f'(a)}{f'(b)}\right|$.

Proof. Since $|f'|^q$ is an $s-$geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1 and well-known Power mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left[\int_0^1 |-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 |t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$

$$\leq \frac{b-a}{4}\left[\left[\int_0^1 tdt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}+\left[\int_0^1 tdt\right]^{1-\frac{1}{q}}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}+\left[\int_0^1\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right]^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t\left(\left|f'(a)^{q\left(\frac{1+t}{2}\right)^s} f'(b)^{q\left(\frac{1-t}{2}\right)^s}\right|\right)dt\right]^{\frac{1}{q}}+\left[\int_0^1\left(t\left|f'(b)^{q\left(\frac{1+t}{2}\right)^s} f'(a)^{q\left(\frac{1-t}{2}\right)^s}\right|\right)^q dt\right]^{\frac{1}{q}}\right].$$

By (8), when $0<|f'(a)|,|f'(b)|\leq 1$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t\left|f'(a)^{sq\left(\frac{1+t}{2}\right)} f'(b)^{sq\left(\frac{1-t}{2}\right)}\right|dt\right]^{\frac{1}{q}}+\left[\int_0^1 t\left|f'(b)^{sq\left(\frac{1+t}{2}\right)} f'(a)^{sq\left(\frac{1-t}{2}\right)}\right|dt\right]^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\mu^{\frac{sq}{2}}\int_0^1 t\mu^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left[\mu^{-\frac{sq}{2}}\int_0^1 t\mu^{-\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\mu^{\frac{s}{2}}\left[\int_0^1 t\left|\frac{f'(a)}{f'(b)}\right|^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[\int_0^1 t\left|\frac{f'(a)}{f'(b)}\right|^{-\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\mu^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

when $\left|f'(b)\right| \leq 1 \leq \left|f'(a)\right|$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t\left|f'(a)^{sq\left(\frac{1+t}{2}\right)+q(1-s)} f'(b)^{sq\left(\frac{1-t}{2}\right)}\right|dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1 t\left|f'(b)^{sq\left(\frac{1+t}{2}\right)} f'(a)^{sq\left(\frac{1-t}{2}\right)+q(1-s)}\right|dt\right]^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\mu^{\frac{sq}{2}}\left|f'(a)\right|^{q(1-s)}\int_0^1 t\mu^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left[\mu^{-\frac{sq}{2}}\left|f'(a)\right|^{q(1-s)}\int_0^1 t\mu^{-\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\mu^{\frac{s}{2}}\left|f'(a)\right|^{(1-s)}\int_0^1 t\mu^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left[\mu^{-\frac{s}{2}}\left|f'(a)\right|^{(1-s)}\left[\int_0^1 t\mu^{-\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'(a)\right|^{(1-s)}\left[\mu^{\frac{s}{2}}\right]\left(\mu^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

when $1 \leq \left|f'(a)\right|, \left|f'(b)\right|$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t\left|f'(a)^{sq\left(\frac{1+t}{2}\right)+q(1-s)} f'(b)^{sq\left(\frac{1-t}{2}\right)+q(1-s)}\right|dt\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\int_0^1 t\left|f'(b)^{sq\left(\frac{1+t}{2}\right)+q(1-s)} f'(a)^{sq\left(\frac{1-t}{2}\right)+q(1-s)}\right|dt\right]^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\mu^{\frac{sq}{2}}\left|f'(a)f'(b)\right|^{q(1-s)}\int_0^1 t\mu^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left[\mu^{-\frac{sq}{2}}\left|f'(a)f'(b)\right|^{q(1-s)}\int_0^1 t\mu^{-\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left[\mu^{\frac{s}{2}}\left|f'(a)f'(b)\right|^{(1-s)}\int_0^1 t\mu^{\frac{sq}{2}t}dt\right]^{\frac{1}{q}}+\left[\mu^{-\frac{s}{2}}\left|f'(a)f'(a)\right|^{(1-s)}\int_0^1 t\mu^{-\frac{sq}{2}t}dt\right]^{\frac{1}{q}}\right]$$

$$=\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left|f'(a)f'(b)\right|^{1-s}\left(\mu^{\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{\frac{1}{q}}+\mu^{-\frac{s}{2}}\left[g_1\left(\alpha\left(\frac{-sq}{2},\frac{-sq}{2}\right)\right)\right]^{\frac{1}{q}}\right),$$

which completes the proof.

Theorem 4. Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a,b \in I$, with $a<b$ and $f' \in L([a,b])$ and $0<|f'(a)|,|f'(b)| \le 1$. If $|f'|$ is $s-$geometrically convex and monotonically decreasing on $[a,b]$ for $\mu_1,\mu_2,\eta_1,\eta_2>0$ with $\mu_1+\eta_1=1$ and $\mu_2+\eta_2=1$ and $s\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}|f'(a)f'(b)|^{\frac{s}{2}} \times\left[\frac{(1+\mu_2)\mu_1^2+(1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)}+\eta_1 g_2\left(\alpha\left(\frac{s}{2\eta_1},\frac{s}{2\eta_1}\right)\right)+\eta_2 g_2\left(\alpha\left(\frac{s}{2\eta_2},\frac{s}{2\eta_2}\right)\right)\right]. \tag{15}$$

Proof. Since $|f'|$ is $s-$geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 1, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1|t|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\
&\le \frac{b-a}{4}\left[\int_0^1|-t|\left|f'\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|dt+\int_0^1|t|\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|dt\right] \\
&\le \frac{b-a}{4}\left[\int_0^1|-t||f'(a)|^{\left(\frac{1+t}{2}\right)^s}|f'(b)|^{\left(\frac{1-t}{2}\right)^s}dt+\int_0^1|t||f'(b)|^{\left(\frac{1+t}{2}\right)^s}|f'(a)|^{\left(\frac{1-t}{2}\right)^s}dt\right].
\end{aligned}$$

When $0<|f'(a)|,|f'(b)|\le 1$, by (8), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{b-a}{4}\left[\int_0^1|-t||f'(a)|^{s\left(\frac{1+t}{2}\right)}|f'(b)|^{s\left(\frac{1-t}{2}\right)}dt+\int_0^1|t||f'(b)|^{s\left(\frac{1+t}{2}\right)}|f'(a)|^{s\left(\frac{1-t}{2}\right)}dt\right] \\
&=\frac{b-a}{4}|f'(a)f'(b)|^{\frac{s}{2}}\left[\int_0^1|-t|\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2}}dt+\int_0^1|t|\left|\frac{f'(a)}{f'(b)}\right|^{-\frac{st}{2}}dt\right],
\end{aligned} \tag{16}$$

for all $t\in[0,1]$. Using well-known inequality $mn\le \mu m^{1/\mu}+\eta n^{1/\eta}$, on the right side of (16), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left[\int_0^1 |-t|\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2}}dt+\int_0^1 |t|\left|\frac{f'(a)}{f'(b)}\right|^{-\frac{st}{2}}dt\right]$$

$$\leq \frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left[\mu_1\int_0^1 |-t|^{\frac{1}{\mu_1}}dt+\eta_1\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2\eta_1}}dt+\mu_2\int_0^1 |t|^{\frac{1}{\mu_2}}dt+\eta_2\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{-\frac{st}{2\eta_2}}dt\right]$$

$$=\frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left[\frac{\mu_1^2}{1+\mu_1}+\eta_1\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{\frac{st}{2\eta_1}}dt+\frac{\mu_2^2}{1+\mu_2}+\eta_2\int_0^1\left|\frac{f'(a)}{f'(b)}\right|^{-\frac{st}{2\eta_2}}dt\right]$$

$$=\frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}\left[\frac{\mu_1^2}{1+\mu_1}+\eta_1 g_2\left(\alpha\left(\frac{s}{2\eta_1},\frac{s}{2\eta_1}\right)\right)+\frac{\mu_2^2}{1+\mu_2}+\eta_2 g_2\left(\alpha\left(\frac{s}{2\eta_2},\frac{s}{2\eta_2}\right)\right)\right]$$

$$=\frac{b-a}{4}\left|f'(a)f'(b)\right|^{\frac{s}{2}}$$

$$\times\left[\frac{(1+\mu_2)\mu_1^2+(1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)}+\eta_1 g_2\left(\alpha\left(\frac{s}{2\eta_1},\frac{s}{2\eta_1}\right)\right)+\eta_2 g_2\left(\alpha\left(\frac{s}{2\eta_2},\frac{s}{2\eta_2}\right)\right)\right],$$

and we get, in here, if $\left|f'(a)\right|=\left|f'(b)\right|=1,$ we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left[\frac{(1+\mu_2)\mu_1^2+(1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)}+\eta_1+\eta_2\right]$$

which the proof is completed.

3. Applications to special means for positive numbers

Let

$$A(a,b) = \frac{a+b}{2},\ L(a,b)=\frac{b-a}{ln\ b-ln\ a}\quad (a\neq b),$$

$$L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(b-a)(p+1)}\right)^{1/p},\ a\neq b,\ p\in\mathbb{R},\ p\neq -1,0$$

be the arithmetic, logarithmic, generalized logarithmic means for $a,b>0$ respectively.

In the following propositions, $\alpha(u,v)=\frac{\left|f'(a)\right|^u}{\left|f'(b)\right|^v}=\frac{\left|a^{s-1}\right|^u}{\left|b^{s-1}\right|^v}.$

Proposition 1. Let $0<a<b\leq 1,$ with $a\neq b,$ and $0<s<1.$ Then, we have

$$\frac{1}{s}\left|A(a^s,b^s)-L_s(a^s,b^s)\right|$$

$$\leq\frac{b-a}{4}(ab)^{\frac{s}{2}(s-1)}\left\{\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\right)^2}+\frac{\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\right)^2}\right.$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1.$

Proof. Let $f(x)=\frac{x^s}{s},\ x\in(0,1],\ 0<s<1,$ then $\left|f'(x)\right|=x^{s-1},\ x\in(0,1]$ is an $s-$geometrically convex mapping. The assertion follows from Theorem 1 applied to $s-$geometrically convex mapping $\left|f'(x)\right|=x^{s-1},\ x\in(0,1].$

Example 2. Let $f(x)=\frac{x^s}{s},\ x\in(0,1],\ 0<s<1,$ then $\left|f'(x)\right|=x^{s-1},\ x\in(0,1]$ is an $s-$geometrically convex mapping. If we apply in Theorem 1, for $s=0.5,\ a=0.89,\ b=0.9,$ we get

$$\frac{1}{s}\left|\frac{a^s+b^s}{2}-\left(\frac{\mathrm{b}^{s+1}-\mathrm{a}^{s+1}}{(b-a)(s+1)}\right)\right|=4.\ 921067116\times 10^{-6}$$

$$\leq\frac{b-a}{4}(ab)^{\frac{s}{2}(s-1)}\left\{\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\right)^2}+\frac{\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}-\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{s}{2}}\right)^2}\right.$$

$$=2.\ 570313847\times 10^{-2}$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1.$ And similarly, if we apply for $s=0.2,\ a=0.15,\ b=0.6,$ we obtain

$$9.\ 780804473\times 10^{-2}\ \leq 0.136819309\ 576863680\ 170486$$

for $s=0.75,\ a=0.45,\ b=0.86,$
we obtain

$$6.\ 115413651\times 10^{-2}\ \leq 0.112144032\ 368736206\ 184243$$

etc.

Proposition 2. Let $0<a<b\leq 1,$ with $a\neq b,$ and $0<s<1,$ and $p,q>1.$ Then, we have

$$\frac{1}{s}\left|A(a^s,b^s)-L_s(a^s,b^s)\right|$$
$$\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left|ab\right|^{\frac{s}{2}(s-1)}\left(a^{\frac{s}{2}(1-s)}+b^{\frac{s}{2}(1-s)}\right)\left[L\left(a^{\frac{sq}{2}(s-1)},b^{\frac{sq}{2}(s-1)}\right)\right]^{\frac{1}{q}}.$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1.$

Proof. The assertion follows from Theorem 2 applied to $s-$geometrically convex mapping $\left|f'(x)\right|=\frac{x^s}{s},\ x\in(0,1].$

Proposition 3. Let $0<a<b\leq 1,$ with $a\neq b$ and $0<s<1$ and $q\geq 1.$ Then, we have

$$\frac{1}{s}\left|A(a^s,b^s)-L_s(a^s,b^s)\right|\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\times$$
$$\left\{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2}}\left[\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}\ln\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}-\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}+1}{\left(\ln\left|\frac{a}{b}\right|^{(s-1)\frac{sq}{2}}\right)^2}\right]^{\frac{1}{q}}+\left|\frac{b}{a}\right|^{(s-1)\frac{s}{2}}\left[\frac{\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}+\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}-1}{\left(\ln\left|\frac{a}{b}\right|^{-(s-1)\frac{sq}{2}}\right)^2}\right]^{\frac{1}{q}}\right.$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1.$

Proof. The assertion follows from Theorem 3 applied to $s-$geometrically convex mapping $\left|f'(x)\right|=\frac{x^s}{s},\ x\in(0,1].$

Proposition 4. Let $0<a<b\leq 1,$ $0<\mu_1,\mu_2,\eta_1,\eta_2$ with $\mu_1+\eta_1=1$ and $\mu_2+\eta_2=1$ and $0<s<1$ and $q\geq 1.$ Then, we have

$$\frac{1}{s}\left|A(a^s,b^s)-L_s(a^s,b^s)\right|$$
$$\leq \frac{b-a}{4}\left|ab\right|^{\frac{s}{2}(s-1)}\left[\frac{(1+\mu_2)\mu_1^2+(1+\mu_1)\mu_2^2}{(1+\mu_1)(1+\mu_2)}+\eta_1\left[\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_1}}-1}{\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_1}}}\right]++\eta_2\left[\frac{\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_1}}-1}{\ln\left|\frac{a}{b}\right|^{(s-1)\frac{s}{2\eta_1}}}\right]\right],$$

where $0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1.$

Proof. The assertion follows from Theorem 4 applied to $s-$geometrically convex mapping $\left|f'(x)\right|=\frac{x^s}{s},\ x\in(0,1].$

4. Conclusion

Inequalities play a significant role in the pure and applied sciences. Inequalities are becoming more and more attractive in mathematics. Recently, especially integral inequalities have become even more attractive through different types of convex function classes. In this paper, we have establish some new Hadamard-type inequalities using elementary well-known inequalities for functions whose first derivatives absolute values are geometrically and s-geometrically convex. And some applications to special means for positive numbers are given. We believe that this work will give a different perspective to the scientists working on the subject.

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Неравенства типа Адамара для s-гео выпуклых функций

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РЕЗЮМЕ

В настоящей работе авторы устанавливают некоторые новые неравенства типа Адамара, используя элементарные известные неравенства для функций, чьи первые производные абсолютные значения геометрически и s-геометрически выпуклые, которые приведены ниже соответственно

$$f\left(x^t y^{1-t}\right) \le \left[f(x)\right]^t \left[f(y)\right]^{1-t}$$

и

$$f\left(x^t y^{1-t}\right) \le \left[f(x)\right]^{t^s} \left[f(y)\right]^{(1-t)^s}$$

где $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$, для некоторых фиксированных $s \in (0,1]$, $x, y \in I \subset \mathbb{R}_+$ и $t \in [0,1]$. Приводятся некоторые приложения к специальным средствам для положительных чисел.

Ключевые слова: Геометрически выпуклое, неравенство Адамара, S-геометрически выпуклое, специальные средства.